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THE CLASSIFICATION OF SPINORS UNDER GSpin_{14} OVER FINITE FIELDS

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ABSTRACT. The spinors of a 14-dimensional vector space V are studied with respect to the group GSpin_{14} of the 14-dimensional vector space V over finite fields $\mathbf{F}_q$. Results are given as follows: (1) the decomposition of the space of spinors into GSpin_{14}-equivalence classes or "orbits" over $\mathbf{F}_q$, (2) the structure of the fixer of GSpin_{14} for each orbit as an $\mathbf{F}_q$-group.

INTRODUCTION

Between 1969 and 1975, Atiyah, Bernstein, and S. I. Gel′fand [1, 2] proved the following theorem: If K is a local field of characteristic zero and $f(x) \in K[x_1, \dots, x_n] - \{0\}$, then $|f|^s$ has a meromorphic continuation to the whole s-plane, where $|f|^s$ is a distribution in K^n, called the "complex power" of $f(x)$ defined as

$$|f|^s(\Phi) = \int_{K^n} |f(x)|_K^s \Phi(x)\, dx,$$

where $|\cdot|_K$ is an absolute value in K, Φ is a Schwartz-Bruhat function, dx is a Haar measure on K^n, and s is a complex parameter restricted to the right-half plane. Furthermore, the candidates for the poles of $|f|^s$ can be written in terms of the roots of the b-function, or the Bernstein-Sato polynomial $b(s)$ of $f(x)$, i.e., if

$$b(s) = \prod_{\lambda>0}(s+\lambda),$$

then the candidates for the poles of $|f|^s$ are $-\lambda, -\lambda-1, -\lambda-2, \dots$. It is also known that the λ's in $b(s) = \prod_{\lambda>0}(s+\lambda)$ are positive rational numbers (cf. Kashiwara [14]). When K is a p-adic local field with q as the cardinality of its residue field, Igusa [6] has shown that $|f|^s$ is a rational function of $t = q^{-s}$. Furthermore, many examples suggest that an intimate relation between the real poles of $|f|^s$ and the roots of $b(s)$, similar to the one in the archimedean case, would also exist in the p-adic case. Recently, Loeser [17] proved that the real poles of $|f|^s$ are the roots of $b(s)$ if $n = 2$.

Partly because the p-adic case of the theory of the complex powers is not as satisfactory as the archimedean case, Igusa began to study a certain complex-valued p-adic integral especially for the "regular prehomogeneous vector space"

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cases (cf. Sato, Kimura, and Shintani [15, 22]. Serre [23] named it "Igusa's local zeta funcion."

A regular prehomogeneous vector space defined over a field F is a triplet (G, ρ, X) in which $X = \mathrm{Aff}^n$ for some n, G is a reductive F-subgroup of $\mathrm{GL}(X) = \mathrm{GL}_n$ acting transitively on the complement Y of an absolutely irreducible F-hypersurface $f(x) = 0$ in X, and ρ is a rational representation of G on X. $f(x)$ is a homogeneous relative G-invariant, i.e., $f(gx) = \nu(g)f(x)$ for every g in G, and is unique up to a factor in $F - \{0\}$. If K is a p-adic completion of a number field, O_K is the ring of integers of K, $\pi_K O_K$ is the maximal ideal of O_K, and q is the cardinality of $O_K/\pi_K O_K$, then the Igusa local zeta function is defined for any $f(x)$ in $K[x_1, \dots, x_n] - \{0\}$ and for $\mathrm{Re}(s) > 0$ as

$$Z(s) = \int_{O_K^n} |f(x)|_K^s \, dx = |f|^s (\Phi_0),$$

where Φ_0 is the characteristic function of $X(O_K)$ and dx is the Haar measure on K^n normalized as $\mathrm{vol}(O_K^n) = 1$. Igusa [6] has shown that $Z(s)$ has a meromorphic continuation to the whole s-plane and is a rational function of $t = q^{-s}$. $Z(s)$ has been computed for twenty of twenty-nine types of irreducible regular prehomogeneous vector spaces (cf. Igusa [12]).

Due to the difficulty in determining the local zeta function for some of the group invariants, Igusa [5, 7–11] developed a series of methods to calculate the local zeta function as the summation of the product of the cardinalities of the group orbits and the corresponding local zeta functions, i.e.,

$$Z(s) = \int_{O_K^n} |f(x)|_K^s \, dx = \sum_{\xi \in R} |G(\mathbf{F}_q) \cdot \bar{\xi}| \int_{\xi + \pi O_K^n} |f(x)|_K^s \, dx.$$

Here R is a subset of O_K^n such that its image $\overline{R}$ in $\mathbf{F}_q^n$ forms a complete set of representatives of $G(\mathbf{F}_q)$-orbits in $\mathbf{F}_q^n$, i.e.,

$$\coprod_{\xi \in R} G(\mathbf{F}_q) \cdot \bar{\xi} = \mathbf{F}_q^n.$$

It is therefore essential to understand the orbital structure of the group G over finite fields. GSpin_{14} is one of the few algebaic groups associated with regular prehomogeneous vector spaces for which the Igusa local zeta function remains unknown. Based on the works of Popov [20] and Kac and Vinberg [13] concerning Spin_{14} over an algebraically closed field of characteristic zero, the following results are obtained in this paper:

(1) the decomposition of the space of spinors into GSpin_{14}-equivalence classes or "orbits" over $\mathbf{F}_q$, and
(2) the structure of the fixer of GSpin_{14} for each orbit as an $\mathbf{F}_q$-group.

One should point out that the conjecture on the relation between the real poles of the Igusa local function and the roots of the Bernstein-Sato polynimial is verified for GSpin_{14} and the candidates for the real poles of the Igusa loca zeta function of GSpin_{14} are determined in [16] by using the above classification. Futhermore, this conjecture has been verified for any reduced irreducible regular prehomogeneous vector spaces by T. Kimura, F. Sato, and X.-W. Zhu [16].

We should also mention that the poles of $Z(s)$ for curves have been closely examined by D. Meuser [18, 19].

1. Preliminaries

If V is a finite $2n$-dimensional vector space over a field k and if $Q(u)$ is a nondegenerate quadratic form on V, we denote by $(\ ,\ )$ the bilinear form associated to Q so that $(u, v) = Q(u+v) - Q(u) - Q(v)$. We denote by C the Clifford algebra of the pair (V, Q) and by $x \to x'$ its canonical antiautomorphism. Then $C = C^+ \oplus C^-$, where C^+ is the space of invariant elements with respect to this antiautomorphism and C^- is the space of anti-invariant elements.

The Clifford group is

$$\widetilde{G}^* = \{s \in C\,;\, s \text{ invertible in } C \text{ and } sVs^{-1} = V\}.$$

The even Clifford group is

$$(\widetilde{G}^*)^+ = \widetilde{G}^* \cap C^+.$$

The spin group is

$$\mathrm{Spin}_{2n} = \widetilde{G} = \{s \in (\widetilde{G}^*)^+\,;\, ss' = 1\}.$$

We denote by C_W the subalgebra of C generated by any subspace W of V. Note that if W is totally isotropic, i.e., if $Q = 0$ on W, C_W is isomorphic to the exterior algebra $\Lambda(W)$ of W.

We shall use ϕ to denote the vector representation of Spin_{2n} which is the restriction of the epimorphism $\phi\colon \widetilde{G}^* \to \mathrm{Aut}(V, Q)$, given by $\phi(s)\cdot v = svs^{-1}$, to Spin_{2n}. This restriction is an epimorphism onto the connected component of identity of $\mathrm{Aut}(V, Q)$ with kernel $\{\pm 1\}$.

Let $V = L + M$, where L, M are maximal totally isotropic subspaces of V. Choose bases $e_1, \dots, e_n$ and $e_{n+1}, \dots, e_{2n}$ of L and M respectively satisfying $(e_i, e_{n+i}) = 1$ for $1 \le i \le n$ and $(e_i, e_j) = 0$ for any other pairs (i, j), $i \le j$. If $s \in \widetilde{G}^*$, we shall define four $n \times n$ matrices α, β, γ, δ by

$$\phi(s)\cdot(e_1 \cdots e_{2n}) = (e_1 \cdots e_{2n}) \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}.$$

Then, $\alpha^t\beta$, $\gamma^t\delta$ are alternating and $\alpha^t\delta + \beta^t\gamma = 1_n$ or, equivalently, ${}^t\alpha\gamma$, ${}^t\beta\delta$ are alternating and ${}^t\alpha\delta + {}^t\gamma\beta = 1_n$. We shall write

$$\phi(s) = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}.$$

C^+ is an enveloping algebra of $\widetilde{G}$ and it is isomorphic to the direct sum of two total matrix algebras of degree 2^{n-1}. The representation ρ of $\widetilde{G}$ in C^+ is called the spin representation of $\widetilde{G}$. It is the direct sum of two irreducible half-spin representations of $\widetilde{G}$ of degree 2^{n-1}. An element of the representation space is called a spinor. We shall make the half-spin representation explicit as follows.

Put $e_L = e_1 \cdots e_n$ and $e_M = e_{n+1} \cdots e_{2n}$. Then Ce_M is a minimal left ideal of C and the correspondence $x \to xe_M$ gives an isomorphism $C_L \to Ce_M$ of these vector spaces. Therefore there exists a unique element y of C_L satisfying $sxe_M = ye_M$ for any $s \in C$ and $x \in C_L$. Let

$$X = (C_L)^+ = C_L \cap C^+ \cong \Lambda^+(L) = \Lambda^+(\mathrm{Aff}^n),$$

i.e., the sum of all even degree homogeneous parts of $\Lambda(L)$. This is a vector space of dimension 2^{n-1} defined over k. By restricting ρ to $\widetilde{G}$, we get a half-spin representation of $\widetilde{G}$ in X.

Let $s_i(\lambda) = \lambda^{-1} + (\lambda - \lambda^{-1})e_i e_{n+i}$ with $\lambda \in G_m$ and $1 \le i \le n$. Then $\lambda \to s_i(\lambda)$ gives a homomorphism $G_m \to \widetilde{G}$. Since the one-parameter subgroups $\{s_i(\lambda)\}$ and $\{s_j(\lambda)\}$ commute, the mapping $(G_m)^n \to \widetilde{G}$ defined by $(t_1, \dots, t_n) \to \prod_{i=1}^n s_i(t_i)$ is a homomorphism. Its image group $\widetilde{T}$ is a maximal torus of $\widetilde{G}$ and its kernel consists of $(\pm 1, \dots, \pm 1)$ with an even number of minus signs.

Let $s_{ij}(\lambda) = 1 + \lambda e_i e_j$ with $\lambda \in G_a$ and $1 \le i \ne j \le 2n$ such that $(e_i, e_j) = 0$. Then the correspondence $\lambda \to s_{ij}(\lambda)$ defines an isomorphism of G_a to its image group P_{ij} in $\widetilde{G}$. These image groups are the one-parameter unipotent subgroups of $\widetilde{G}$ corresponding to the $2n(n-1)$ roots of $\widetilde{G}$ relative to $\widetilde{T}$.

If W is a subspace of V and if $\Lambda^2(W)$ denotes the homogeneous part of degree two in the exterior algebra C_W, we get a well-defined homomorphism $\exp\colon \Lambda^2(W) \to \widetilde{G}$ such that $\exp(x) = 1 + x + \frac{1}{2!}x^2 + \cdots$ for every x in $\Lambda^2(W)$. The element $\exp(x)$ is called the exponential of x.

Let

$$G = \mathrm{GSpin}_{2n} = (\widetilde{G}^*)^+.$$

Then

$$g_i(t) = t^{1/2} s_i(t)^{1/2} = 1 + (t-1)e_i e_{n+i}, \qquad 1 \le i \le n,$$

are elements of GSpin_{2n} for any $t \in G_m$ and

$$T = \left\{ t = \prod_{i=1}^n g_i(t_i);\ t_1, \dots, t_n \in G_m \right\}$$

is a maximal torus of G. We give the following easily proved lemma for later use.

Lemma. *Let* $1 \le i, j \le 7$. *Then*

$$\phi(s_{i,j}(\lambda)) \cdot e_k = e_k, \qquad k \ne n+i, n+j;$$
$$\phi(s_{i,j}(\lambda)) \cdot e_{n+i} = e_{n+i} - \lambda e_j;$$
$$\phi(s_{i,j}(\lambda)) \cdot e_{n+j} = e_{n+j} + \lambda e_i;$$
$$\phi(s_{n+i,n+j}(\lambda)) \cdot e_k = e_k, \qquad k \ne i, j;$$
$$\phi(s_{n+i,n+j}(\lambda)) \cdot e_i = e_i - \lambda e_{n+j};$$
$$\phi(s_{n+i,n+j}(\lambda)) \cdot e_j = e_j + \lambda e_{n+i};$$
$$\phi(s_{i,n+j}(\lambda)) \cdot e_k = e_k, \qquad k \ne n+i, j;$$
$$\phi(s_{i,n+j}(\lambda)) \cdot e_{n+i} = e_{n+i} - \lambda e_{n+j};$$
$$\phi(s_{i,n+j}(\lambda)) \cdot e_j = e_j + \lambda e_i,$$

where $\lambda \in G_a$. For any $t_1, \ldots, t_7 \in G_m$ we have

$$\phi\left(\prod_{i=1}^{7} g_i(t_i)\right) \cdot e_k = \begin{cases} t_k e_k, & \text{if } 1 \le k \le 7, \\ t_k^{-1} e_k, & \text{if } 8 \le k \le 14, \end{cases}$$

$$\rho\left(\prod_{i=1}^{7} g_i(t_i)\right) \cdot (e_{i_1} \cdots e_{i_p}) = (t_{i_1} \cdots t_{i_p}) e_{i_1} \cdots e_{i_p}.$$

We shall use G_x to denote the fixer of G at the point $x \in X$ with respect to ρ, G_x^0 to denote the connected component of the identity of G_x, and $Z(H)$ to denote the center of the group H. We define

$$e^*_{i_1 \cdots i_p} = \text{the partial products of } e_1, \ldots, e_n \text{ satisfying } e_{i_1} \cdots e_{i_p} e^*_{i_1 \cdots i_p} = e_L.$$

In particular,

$$e_i^* = (-1)^{i-1} e_1 \cdots \hat{e}_i \cdots e_n,$$
$$e_{ij}^* = (-1)^{i+j-1} e_1 \cdots \hat{e}_i \cdots \hat{e}_j \cdots e_n,$$
$$e_{ijk}^* = (-1)^{i+j+k} e_1 \cdots \hat{e}_i \cdots \hat{e}_j \cdots \hat{e}_k \cdots e_n.$$

In the sequel we shall assume $n = 7$, $X = \Lambda^+(\mathrm{Aff}^7)$, and $G = \mathrm{GSpin}_{14}$ unless the contrary is expressly stated.

2. Orbital decomposition

In this section we obtain the representatives of GSpin_{14}-orbits in $\Lambda^+(\mathrm{Aff}^7)$ over any finite field k of characteristic different from 2. Since most of the methods used in this section are the same as Popov's [20] for obtaining the representatives of GSpin_{14}-orbits in $\Lambda^+(\mathrm{Aff}^7)$ over algebraically closed fields of characteristic zero, we shall only sketch the procedure and examine those cases that lead to different results.

We shall say that elements in X are G-equivalent if they lie in the same orbit of G in X, i.e., $\rho(s) \cdot x = y$ for some $s \in G$, denoted by $x \xrightarrow{s} y$. For every element $u = \sum_{1 \le i < j \le 6} \alpha_{ij} e_{ij}^*$, $\alpha_{ij} \in k$, in $\Lambda^4(\mathrm{Aff}^6)$, the rank of the alternating matrix (α_{ij}) is called the rank of the spinor u and denoted as $\operatorname{rank}(u)$.

For any nonzero spinor $x \in X$, $x = x_0 + x_2 + x_4 + x_6$, where $x_0 \in k^*$, $x_i \in \Lambda^i(\mathrm{Aff}^7)$ for $i = 2, 4, 6$. By Lemma 1 of [4], have

$$x \xrightarrow{s_1(x_0)} 1 + x_2' + x_4' + x_6' \xrightarrow{\exp(-x_2')} 1 + x_4'' + x_6''.$$

Therefore, we can assume that $x = 1 + x_4 + x_6$. We shall study spinors of this type according to $x_6 \ne 0$ and $x_6 = 0$.

From Lemma 1 of [4], have

$$G_1 \cong \phi(G_1) = \left\{ \begin{bmatrix} \alpha & 0 \\ \gamma & \delta \end{bmatrix},\ \delta = {}^t\alpha^{-1} \in SL_7,\ {}^t\alpha\gamma \in \mathrm{Alt}_7 \right\}.$$

Let H be the subgroup of G_1 such that

$$\phi(H) = \left\{ \begin{bmatrix} \alpha & 0 \\ 0 & {}^t\alpha^{-1} \end{bmatrix},\ \alpha \in SL_7 \right\},$$

let S be the subgroup of H for which

$$\alpha = \begin{bmatrix} \varepsilon & 0 \\ 0 & \det \varepsilon^{-1} \end{bmatrix}$$

with $\varepsilon \in GL_6$ in $\phi(S)$, and let $\widetilde{S}$ be the subgroup of S for which $\varepsilon \in SL_6$ in $\phi(\widetilde{S})$. Thus $H \cong SL_7$, $S \cong GL_6$, and $\widetilde{S} \cong SL_6$.

First, consider the case when $x_6 \neq 0$. Then

$$x = 1 + x_4 + x_6 \xrightarrow{h \in H} 1 + x_4 + e_7^* .$$

If $x_4 = 0$, then

$$x = 1 + e_7^*. \tag{1}$$

If $x_4 \neq 0$, write x_4 in the form $x_4 = ye_7 + z$, where $y \in \Lambda^3(\mathrm{Aff}^6)$, $z \in \Lambda^4(\mathrm{Aff}^6)$. With respect to the group S, it is possible to bring x into one of the forms

$$1 + ye_7 + e_7^*, \tag{2}$$

$$1 + ye_7 + e_{147}^* + e_7^*, \tag{3}$$

$$1 + ye_7 + e_{147}^* + e_{257}^* + e_7^*, \quad \text{or} \tag{4}$$

$$1 + ye_7 + e_{147}^* + e_{257}^* + e_{367}^* + pe_7^*, \qquad p \in k^*, \tag{5}$$

depending on whether $\operatorname{rank}(z) = 0, 2, 4$, or 6, where $y \in \Lambda^3(\mathrm{Aff}^6)$.

As in [20], we may claim that spinors of type (3) and (4) are equivalent to spinors of type (2). Futhermore, considering the action of S on spinors of type (2) and the fact that all the GL_6-inequivalent trivectors of six-dimensional space are those GL_7-inequivalent trivectors of seven-dimensional space involving six vectors, and by the classification of trivectors of seven-dimensional space over finite fields [11], it follows that y can be transformed to one of the pairwise S-inequivalent forms

$$\begin{aligned}
&e_1e_2e_3, \quad e_1e_2e_3 + e_4e_5e_6, \quad e_1e_2e_3 + e_3e_4e_5, \\
&e_1e_2e_3 + e_4e_3e_5 + e_6e_5e_2, \quad \text{or} \\
&4^{-1}e_{123}^* + (4\lambda)^{-1}(e_{156}^* - e_{246}^* + e_{345}^*), \qquad \lambda \in \mathbf{F}_q^* - (\mathbf{F}_q^*)^2.
\end{aligned}$$

Therefore, with the action of S, x can be transformed into one of the following:

$$\begin{aligned}
&1 + \nu e_1e_2e_3e_7 + \nu^{-1}e_7^*, \\
&1 + \nu e_1e_2e_3e_7 + \nu e_4e_5e_6e_7 + \nu^{-1}e_7^*, \\
&1 + \nu(4^{-1}e_{123}^* + (4\lambda)^{-1}(e_{156}^* - e_{246}^* + e_{345}^*)) + \nu^{-1}e_7^*, \\
&1 + \nu e_1e_2e_3e_7 + \nu e_3e_4e_5e_7 + \nu^{-1}e_7^*, \quad \text{or} \\
&1 + \nu e_1e_2e_3e_7 + \nu e_4e_3e_5e_7 + \nu e_6e_5e_2e_7 + \nu^{-1}e_7^*,
\end{aligned}$$

where $\nu \in k^*$. Applying the appropriate $h \in H$ such that $\phi(h)e_i = \lambda_i e_i$ with $\lambda_i \in k^*$, $1 \le i \le 6$, to the above spinors, we conclude that spinors of type (2) are equivalent to one of the following spinors :

$$1 + e_1e_2e_3e_7 + e_7^*, \tag{6}$$

$$p(1 + e_1e_2e_3e_7 + e_4e_5e_6e_7 + e_7^*), \qquad p \in k^*, \tag{7}$$

$$p(1 + 4^{-1}e_{123}^* + (4\lambda)^{-1}(e_{156}^* - e_{246}^* + e_{345}^*) - e_7^*), \qquad p \in k^*, \tag{8}$$

$$1 + e_1e_2e_3e_7 + e_3e_4e_5e_7 + e_7^*, \quad \text{or} \tag{9}$$

$$1 + e_1e_2e_3e_7 + e_4e_3e_5e_7 + e_6e_5e_2e_7 + e_7^*, \tag{10}$$

where $\lambda \in \mathbf{F}_q^* - (\mathbf{F}_q^*)^2$, with q a power of any odd number.

Now we consider the spinor of type (5). Let R be the subgroup of $\widetilde{S}$ such that

$$\phi(R) = \left\{ \begin{bmatrix} A & 0 \\ 0 & {}^tA^{-1} \end{bmatrix} ; A = \begin{bmatrix} \varepsilon & 0 \\ 0 & 1 \end{bmatrix}, \varepsilon \in SP_6 \right\}.$$

Then $R \cong SP_6$. By Lemma 5 of [4], $R \subset G_{a_0}$ for $a_0 = e_1e_4 + e_2e_5 + e_3e_6$. It is also clear that $\Lambda^p(\mathrm{Aff}^7)$, $\Lambda^p(\mathrm{Aff}^6)$, and spinor e_7 remain invariant under R. Hence , spinors of type (5), $x = 1 + ye_7 + e_{147}^* + e_{257}^* + e_{367}^* + pe_7^*$, are R-inequivalent if and only if $y \in \Lambda^3(\mathrm{Aff}^6)$ are SP_6-inequivalent. We shall first consider the classification of the trivector of six-dimensional space with respect to SP_6.

Theorem 1. *Every trivector in $\Lambda^3(\mathrm{Aff}^6)$ is SP_6-equivalent to one of the following twenty pairwise SP_6-inequivalent trivectors:*

(a) 0;

(b) $e_1e_2e_3$;

(c) $e_1e_2e_5 + e_1e_3e_6$;

(d) $e_1e_2e_3 + pe_4e_5e_6$, $p \in k^*$;

(e) $e_1e_4e_3 + e_5e_2e_3 + p(e_3e_1e_4 + e_3e_2e_5)$, $p \in k^*$;

(f) $e_1e_2e_3 + pe_4e_5e_6 + e_4e_2e_5 + e_4e_3e_6$, $p = 0$, $p \in k^*$;

(g) $e_1e_4e_3 + e_5e_2e_3 + e_1e_2e_6 + e_4e_2e_5 + e_4e_3e_6$;

(h) $e_1e_2e_3 + pe_4e_5e_6 + e_1e_2e_5 + e_1e_3e_6 + e_5e_1e_4 + e_5e_3e_6$, *with* $p \in k^*$;

(i) $e_1e_4e_3 + e_5e_2e_3 + e_1e_2e_5 + e_1e_3e_6 + p(e_2e_1e_4 + e_2e_3e_6)$, *with* $p = 0$, $p = 1$;

(j) $e_1e_4e_3 + e_5e_2e_3 + pe_1e_2e_6 + r(e_6e_1e_4 + e_6e_2e_5)$, *with*

$$\begin{cases} p = 0 \\ r = 0 \end{cases}, \quad \begin{cases} p = 0 \\ r \in k^* \end{cases}, \quad \begin{cases} p = 1 \\ r = 0 \end{cases}, \quad \begin{cases} p = 1 \\ r \in k^* \end{cases};$$

(k) $e_1e_2e_3 + re_4e_5e_6 + e_1e_2e_5 + e_1e_3e_6 + p(e_4e_2e_5 + e_4e_3e_6)$, *with*

$$\begin{cases} p = 0 \\ r = 0 \end{cases}, \begin{cases} p = 0 \\ r \in k^* \end{cases}, \begin{cases} p \in k^* \\ r \in k^* \end{cases};$$

(l) $2^{-1}e_2e_3e_4 + (2\lambda)^{-1}e_1e_3e_5$;

(m) $2^{-1}e_2e_3e_4 + (2\lambda)^{-1}e_1e_3e_5 + 2e_1e_2e_5 + 2e_1e_3e_6$,

where $\lambda \in \mathbf{F}_q^* - (\mathbf{F}_q^*)^2$, with q a power of any odd prime number.

Proof. The first eighteen pairwise SP_6-inequivalent trivectors (a)–(k) are obtained in the same way as in [20]. However the fixers of $\xi = e_1e_4e_3 + e_5e_2e_3$ and $\eta = e_1e_4e_3 + e_5e_2e_3 + e_1e_2e_5 + e_1e_3e_6 + e_2e_1e_4 + e_2e_3e_6$ in SP_6 have two connected components, and the representatives ξ' and η' of the other connected components are the elements of SP_6 defined by $e_1 \to e_2$, $e_2 \to e_1$, $e_3 \to -e_3$, i.e.,

$$\phi(\xi'), \phi(\eta') = \begin{bmatrix} \alpha & 0 \\ 0 & {}^t\alpha^{-1} \end{bmatrix}, \qquad \alpha = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

Take λ from $\mathbf{F}_q^* - (\mathbf{F}_q^*)^2$, and put $\mu = \lambda^{1/2}$ so that $\mu^\sigma = -\mu$ for $\sigma \in \mathrm{Gal}(\mathbf{F}_{q^2}/\mathbf{F}_q)$. Choose $g \in SP_6(\mathbf{F}_{q^2})$ such that

$$g = \begin{bmatrix} g_1 & 0 \\ 0 & {}^tg_1^{-1} \end{bmatrix}, \qquad g_1 = \begin{bmatrix} 1 & 1 & 0 \\ \mu & -\mu & 0 \\ 0 & 0 & -(2\mu)^{-1} \end{bmatrix} \in SL_3,$$

and $\gamma \in G_\xi$ such that

$$\phi(\gamma) = \begin{bmatrix} \alpha & 0 \\ 0 & {}^t\alpha^{-1} \end{bmatrix}, \qquad \alpha = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

Then $G_\xi = G_\xi^0 \amalg \gamma G_\xi^0$, $\gamma^2 = 1$. By a theorem of Igusa's [11], we have $g^\sigma = g\gamma$ and

$$\begin{aligned}
\xi' &= g \cdot (e_1e_4e_3 + e_5e_2e_3) \\
&= (e_1 + \mu e_2)(2^{-1}e_4 + (2\mu)^{-1}e_5)(-(2\mu)^{-1}e_3) \\
&\quad + (2^{-1}e_4 - (2\mu)^{-1}e_5)(e_1 - \mu e_2)(-(2\mu)^{-1}e_3) \\
&= 2^{-1}e_2e_3e_4 + (2\lambda)^{-1}e_1e_3e_5, \\
\eta' &= g \cdot (e_1e_4e_3 + e_5e_2e_3 + e_1e_2e_5 + e_1e_3e_6 + e_2e_1e_4 + e_2e_3e_6) \\
&= 2^{-1}e_2e_3e_4 + (2\lambda)^{-1}e_1e_3e_5 + (e_1 + \mu e_2 + e_1 - \mu e_2)(-(2\mu)^{-1}e_3)(-2\mu e_6) \\
&\quad + (e_1 + \mu e_2)(e_1 - \mu e_2)(2^{-1}e_4 - (2\mu)^{-1}e_5 - 2^{-1}e_4 - (2\mu)^{-1}e_5) \\
&= 2^{-1}e_2e_3e_4 + (2\lambda)^{-1}e_1e_3e_5 + 2e_1e_2e_5 + 2e_1e_3e_6,
\end{aligned}$$

which are the trivectors (l) and (m).

Now if the factor y in spinor (5) has one of the forms in Theorem 1, other than type (a), (b), and (l), then it is equivalent to the spinors of the form

$$1 + ye_7 + z + pe_7^*$$

with $p \in k^*$, $y \in \Lambda^3(\mathrm{Aff}^6)$, $z \in \Lambda^4(\mathrm{Aff}^6)$, and $\mathrm{rank}(z) < 6$ by the action of $h \in H$ such that $\phi(h)e_i = e_i$ for $1 \leq i \leq 6$, and

$$\phi(h)e_7 = \begin{cases} e_4 + e_7, & \text{if } y = \text{types (c), (i), (k);} \\ -e_1 + e_7, & \text{if } y = \text{types (f), (g);} \\ -e_2 + e_7, & \text{if } y = \text{type (h);} \\ e_6 + e_7, & \text{if } y = \text{type (j);} \\ -p^{-1}e_2 + e_5 + e_7, & \text{if } y = \text{type (d);} \\ (p+1)^{-1}e_6 + e_7, & \text{if } y = \text{type (e) and } p \neq -1; \\ -2^{-1}e_6 + e_7, & \text{if } y = \text{type (e) and } p = -1; \\ 2^{-1}e_4 + e_7, & \text{if } y = \text{type (m).} \end{cases}$$

Consequently, it is equivalent to one of the spinors of types (6)–(10). Otherwise we have the following situation:

(a) Spinors of type (5) with $y = 0$ are equivalent to the spinor

$$1 + e_{147}^* + e_{257}^* \tag{11}$$

when $p = \pm 2$, and

$$1 + \left(\sqrt{p^2 - 4}\right)e_7^* \tag{12}$$

when $p \neq \pm 2$, over $k(\sqrt{p^2 - 4})$.

(b) Spinors of type (5) with $y = e_1e_2e_3$ are equivalent to the spinor

$$1 + e_1e_2e_3e_7 + e_{147}^* + e_{257}^* \tag{13}$$

when $p = \pm 2$, and

(14) $$1 + e_1e_2e_3e_7 + (\sqrt{p^2-4})e_7^*$$

when $p \neq \pm 2$, over $k(\sqrt{p^2-4})$.

(l) Spinors of type (5) with $y = 2^{-1}e_2e_3e_4 + (2\lambda)^{-1}e_1e_3e_5$, $\lambda \in \mathbf{F}_q^* - (\mathbf{F}_q^*)^2$, are equivalent to spinor (9) over $k(\sqrt{p^2-4})$ when $p \neq \pm 2$, and

(15) $$1 + 4^{-1}e_{123}^* + (4\lambda)^{-1}(e_{156}^* - e_{246}^* + e_{345}^*)$$

when $p = \pm 2$.

Proof. (a), (b) Let $\varepsilon = p/2$. Then

$$\begin{aligned}&1 + e_{147}^* + e_{257}^* + e_{367}^* + pe_7^*\\ &\xrightarrow{s_{1,4}(-\varepsilon)s_{2,5}(-\varepsilon)s_{10,13}(-\varepsilon)} 1 + e_{147}^* + e_{257}^* + \left(1 - \frac{p^2}{4}\right)e_{367}^*\\ &= 1 + e_{147}^* + e_{257}^*, \quad \text{when } p = \pm 2,\end{aligned}$$

and

$$\begin{aligned}&1 + e_1e_2e_3e_7 + e_{147}^* + e_{257}^* + e_{367}^* + pe_7^*\\ &\xrightarrow{s_{1,4}(-\varepsilon)s_{2,5}(-\varepsilon)s_{10,13}(-\varepsilon)} 1 + e_1e_2e_3e_7 + e_{147}^* + e_{257}^* + \left(1 - \frac{p^2}{4}\right)e_{367}^*\\ &= 1 + e_1e_2e_3e_7 + e_{147}^* + e_{257}^*, \quad \text{when } p = \pm 2.\end{aligned}$$

When $p \neq \pm 2$, let $\omega = (p - \sqrt{p^2-4})/2$. Then

$$\begin{aligned}&1 + e_{147}^* + e_{257}^* + e_{367}^* + pe_7^*\\ &\xrightarrow{s_{9,12}(\omega)s_{8,11}(\omega)} (1-\omega^2) + \omega(2-\omega p)e_3e_6 + \omega(e_1e_4 + e_2e_5)\\ &\qquad + (1-\omega p)(e_{147}^* + e_{257}^*) + e_{367}^* + pe_7^*\\ &\xrightarrow{s_1(1-\omega^2)} 1 + \omega(2-\omega p)(1-\omega^2)e_3e_6 + \omega(1-\omega^2)^{-1}e_2e_5\\ &\qquad + \omega(1-\omega^2)e_1e_4 + (1-\omega p)(1-\omega^2)^{-1}e_{147}^*\\ &\qquad + (1-\omega p)(1-\omega^2)e_{257}^* + (1-\omega^2)e_{367}^* + p(1-\omega^2)e_7^*\\ &\xrightarrow{\exp(-\omega(2-\omega p)(1-\omega^2)^{-1}e_3e_6)} 1 + \omega(1-\omega^2)^{-1}e_2e_5 + \omega(1-\omega^2)e_1e_4\\ &\qquad + \left[\frac{1-\omega p}{1-\omega^2} + \frac{\omega^2(2-\omega p)}{(1-\omega^2)^2}\right]e_{147}^*\\ &\qquad + [(1-\omega p)(1-\omega^2) + \omega^2(2-\omega p)]e_{257}^*\\ &\qquad + (1-\omega^2)e_{367}^* + (p-2\omega)e_7^*\\ &= 1 + \omega(1-\omega^2)^{-1}e_2e_5 + \omega(1-\omega^2)e_1e_4 + (1-\omega^2)e_{367}^* + (p-2\omega)e_7^*\\ &\xrightarrow{\exp(-\omega(1-\omega^2)^{-1}e_2e_5 - \omega(1-\omega^2)e_1e_4)} 1 + \omega_1e_{367}^* + \omega_2e_7^*,\\ &\qquad\qquad \omega_1 = 1 + \omega^2,\ \omega_2 = p - 2\omega = \sqrt{p^2-4}\\ &\xrightarrow{s_{10,13}(\omega_1\omega_2^{-1})} 1 + (\sqrt{p^2-4})e_7^*.\end{aligned}$$

Applying the same procedure as above to $1 + e_1e_2e_3e_7 + e_{147}^* + e_{257}^* + e_{367}^* + pe_7^*$, followed by an application of the action of $h \in H$ such that $\phi(h)e_i = e_i$ for

$i \neq 3, 6$, $\phi(h)e_3 = (1-\omega^2)^{-1}e_3$, and $\phi(h)e_6 = (1-\omega^2)e_6$ yields $1+e_1e_2e_3e_7+(\sqrt{p^2-4})e_7^*$.

(l) Let $\varepsilon = p/2$. Then

$$\begin{aligned}1 + 2^{-1}e_2e_3e_4e_7 + (2\lambda)^{-1}e_1e_3e_5e_7 + e_{147}^* + e_{257}^* + e_{367}^* + pe_7^*\\ \xrightarrow{s_{1,4}(-\varepsilon)s_{2,5}(-\varepsilon)s_{10,13}(-\varepsilon)} 1 + 2^{-1}e_2e_3e_4e_7 + (2\lambda)^{-1}e_1e_3e_5e_7\\ + e_{147}^* + e_{257}^* + \left(1 - \frac{p^2}{4}\right)e_{367}^*.\end{aligned}$$

When $p = \pm 2$, consider $h \in H$ such that

$$\phi(h)e_1 = -e_1, \quad \phi(h)e_2 = -e_4, \quad \phi(h)e_3 = (8\lambda)^{-1}e_7, \quad \phi(h)e_4 = e_2,$$
$$\phi(h)e_5 = \lambda e_5, \quad \phi(h)e_6 = -2e_6, \quad \phi(h)e_7 = -4e_3.$$

Then

$$\begin{aligned}1 + 2^{-1}e_2e_3e_4e_7 + (2\lambda)^{-1}e_1e_3e_5e_7 + e_{147}^* + e_{257}^*\\ \xrightarrow{h} 1 + 4^{-1}e_{123}^* + (4\lambda)^{-1}(e_{156}^* - e_{246}^* + e_{345}^*).\end{aligned}$$

When $p \neq \pm 2$, apply the same action which we have used in (a) and (b) followed by the action of $h \in H$ such that

$$\phi(h)e_1 = -2\lambda(1-\omega^2)^{-1}e_4, \qquad \phi(h)e_2 = 2(1-\omega^2)e_2,$$
$$\phi(h)e_3 = \left(\sqrt{p^2-4}\right)^{-1}e_3, \quad \phi(h)e_4 = e_1, \quad \phi(h)e_5 = e_5,$$
$$\phi(h)e_6 = (4\lambda)^{-1}e_6, \qquad \phi(h)e_7 = \left(\sqrt{p^2-4}\right)e_7.$$

We obtain

$$\begin{aligned}1 + \frac{(1-\omega^2)^{-1}}{2}e_2e_3e_4e_7 + \frac{(1-\omega^2)}{2\lambda}e_1e_3e_5e_7 + \left(\sqrt{p^2-4}\right)e_7^*\\ \xrightarrow{h} 1 + e_1e_2e_3e_7 + e_3e_4e_5e_7 + e_7^*.\end{aligned}$$

Futhermore, if $\sqrt{p^2-4} \in k^*$, with the action of $h \in H$ such that $\phi(h)e_i = e_i$ for $i \neq 3, 7$, and $\phi(h)e_3 = (\sqrt{p^2-4})^{-1}e_3$, and $\phi(h)e_7 = (\sqrt{p^2-4})e_7$, then spinors (12) and (14) are equivalent to spinors (1) and (6), respectively.

If $\sqrt{p^2-4} \notin k^*$, spinor (12) is equivalent to

$$1 + e_{147}^* + e_{257}^* - \frac{\lambda}{4}e_{367}^* \tag{16}$$

as shown above and spinor (14) is equivalent to spinor (15) by the action of

$$h_4h_3s_{5,6}(-1)h_2s_{11,14}(-1)s_{12,13}(-1)s_{4,7}(-1)h_1,$$

where $h_i \in H$, $i = 1, 2, 3, 4$, such that

$$\phi(h_i) = \begin{bmatrix} \alpha_i & 0 \\ 0 & {}^t\alpha^{-1} \end{bmatrix},$$

with

$$\alpha_1 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\alpha_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -(\sqrt{p^2-4})^{-1} & 0 & 0 & 1 \end{bmatrix},$$

$$\alpha_3 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & \sqrt{p^2-4} \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\alpha_4 = \begin{bmatrix} I_3 & I_3 & 0 \\ \sqrt{p^2-4}I_3 & -\sqrt{p^2-4}I_3 & 0 \\ 0 & 0 & -(2\sqrt{p^2-4})^{-3} \end{bmatrix}.$$

Consolidating these results, we have

Proposition. *Every spinor of the form* $1 + x_4 + x_6$ *with* $x_6 \neq 0$ *is equivalent to one of the following spinors:*

(1) $1 + e_7^*$,

(16) $1 + e_{147}^* + e_{257}^* - \frac{\lambda}{4} e_{367}^*$,

(6) $1 + e_1e_2e_3e_7 + e_7^*$,

(15) $1 + 4^{-1}e_{123}^* + (4\lambda)^{-1}(e_{156}^* - e_{246}^* + e_{345}^*)$,

(7) $p(1 + e_1e_2e_3e_7 + e_4e_5e_6e_7 + e_7^*)$,

(8) $p(1 + 4^{-1}e_{123}^* + (4\lambda)^{-1}(e_{156}^* - e_{246}^* + e_{345}^*) - e_7^*)$,

(9) $1 + e_1e_2e_3e_7 + e_3e_4e_5e_7 + e_7^*$,

(10) $1 + e_1e_2e_3e_7 + e_4e_3e_5e_7 + e_6e_5e_2e_7 + e_7^*$,

(11) $1 + e_{147}^* + e_{257}^*$,

(13) $1 + e_1e_2e_3e_7 + e_{147}^* + e_{257}^*$,

where $p \in k^*$ *and* $\lambda \in \mathbf{F}_q^* - (\mathbf{F}_q^*)^2$, *with* q *a power of any odd prime number.*

Now we consider the case when $x_6 = 0$. Let

$$\sigma\colon \Lambda^3(\mathrm{Aff}^7) \to \Lambda^4(\mathrm{Aff}^7)$$

be the k-linear mapping with respect to the basis $e_{i_1}e_{i_2}e_{i_3}$, $1 \le i_1 < i_2 < i_3 \le 7$, such that $\sigma(e_{i_1}e_{i_2}e_{i_3}) = e_{i_1i_2i_3}^*$. Since the action of SL_7 on $\Lambda^3(\mathrm{Aff}^7)$ and $\Lambda^4(\mathrm{Aff}^7)$ are contragredient and since

$$\begin{array}{ccc} \Lambda^3(\mathrm{Aff}^7) & \xrightarrow{h\in SL_7} & \Lambda^3(\mathrm{Aff}^7) \\ \Big\downarrow{\scriptstyle\sigma} & & \Big\downarrow{\scriptstyle\sigma} \\ \Lambda^4(\mathrm{Aff}^7) & \xrightarrow{{}^th^{-1}\in SL_7} & \Lambda^4(\mathrm{Aff}^7) \end{array}$$

commutes, any two elements $u, v \in \Lambda^3(\mathrm{Aff}^7)$ are SL_7-equivalent if and only if $\sigma(u), \sigma(v) \in \Lambda^4(\mathrm{Aff}^7)$ are SL_7-equivalent. Using Igusa's [11] classification of the trivectors of seven-dimensional space over finite fields, every element of $\Lambda^3(\mathrm{Aff}^7)$ is SL_7-equivalent to one of the following trivectors:

$$
\begin{aligned}
&0; \quad e_5e_6e_7; \quad e_1e_3e_7 + e_2e_4e_7;\\
&e_1e_4e_7 + e_2e_5e_7 + e_3e_6e_7;\\
&e_2e_3e_4 + e_3e_1e_5 + e_1e_2e_6;\\
&e_1e_2e_3 + e_4e_5e_6;\\
&e_1e_2e_3 + (e_1e_4 + e_2e_5 + e_3e_6)e_7;\\
&e_1e_2e_3 + e_4e_5e_6 + e_1e_4e_7;\\
&e_2e_3e_4 + e_3e_1e_5 + e_1e_2e_6 + (e_1e_4 + e_2e_5 + e_3e_6)e_7;\\
&e_1e_2e_3 + e_4e_5e_6 + (e_1e_4 + e_2e_5 + e_3e_6)e_7;\\
&e_1e_2e_3 + \lambda(e_1e_4e_7 + e_5e_6e_1 + e_6e_4e_2 + e_4e_5e_3); \text{ or}\\
&e_1e_2e_3 + \lambda(e_5e_6e_1 + e_6e_4e_2 + e_4e_5e_3)
\end{aligned}
$$

with $\lambda \in \mathbf{F}_q^* - (\mathbf{F}_q^*)^2$. We conclude that the spinors of the form $x = 1 + x_4$ are equivalent to one of the following spinors:

$$1; \tag{17}$$
$$1 + e_1e_2e_3e_4; \tag{18}$$
$$1 + e_{137}^* + e_{247}^*; \tag{19}$$
$$1 + e_{147}^* + e_{257}^* + e_{367}^*; \tag{20}$$
$$1 + e_{234}^* - e_{135}^* + e_{126}^*; \tag{21}$$
$$1 + e_{123}^* + e_{456}^*; \tag{22}$$
$$1 + e_{123}^* + e_{147}^* + e_{257}^* + e_{367}^*; \tag{23}$$
$$1 + e_{123}^* + e_{456}^* + e_{147}^*; \tag{24}$$
$$1 + e_{234}^* - e_{135}^* + e_{126}^* + e_{147}^* + e_{257}^* + e_{367}^*; \tag{25}$$
$$1 + e_{123}^* + e_{456}^* + e_{147}^* + e_{257}^* + e_{367}^*; \tag{26}$$
$$1 + e_{123}^* + \lambda(e_{147}^* + e_{156}^* - e_{246}^* + e_{345}^*); \text{ or} \tag{27}$$
$$1 + e_{123}^* + \lambda(e_{156}^* - e_{246}^* + e_{345}^*), \tag{28}$$

where $\lambda \in \mathbf{F}_q^* - (\mathbf{F}_q^*)^2$, with q a power of any odd prime number.

As in [20] we may claim that the spinors (19)–(26) are equivalent to spinors of the Proposition. Futhermore, spinors (27) and (28) are equivalent to spinors of the type $1 + x_4 + x_6$, $x_6 \neq 0$, with the action of $s_{5,6}(1)s_{2,3}(\lambda)s_{11,14}(1)$. Consequently they are also equivalent to spinors of the Proposition.

We present the orbital decomposition:

Theorem 2. *Put*

$$
\begin{aligned}
\xi_0 &= p(1 + e_1e_2e_3e_7 + e_4e_5e_6e_7 + e_7^*),\\
\xi_0' &= p(1 + 4^{-1}e_{123}^* + (4\lambda)^{-1}(e_{156}^* - e_{246}^* + e_{345}^*) - e_7^*),\\
\xi_1 &= 1 + e_1e_2e_3e_7 + e_4e_3e_5e_7 + e_6e_5e_2e_7 + e_7^*,\\
\xi_5 &= 1 + e_1e_2e_3e_7 + e_3e_4e_5e_7 + e_7^*,\\
\xi_{10} &= 1 + e_1e_2e_3e_7 + e_7^*,
\end{aligned}
$$

$$\begin{aligned}
\xi_{10}' &= 1 + 4^{-1}e_{123}^* + (4\lambda)^{-1}(e_{156}^* - e_{246}^* + e_{345}^*),\\
\xi_{14} &= 1 + e_1e_2e_3e_7 + e_{147}^* + e_{257}^*,\\
\xi_{20} &= 1 + e_7^*,\\
\xi_{20}' &= 1 + e_{147}^* + e_{257}^* - \frac{\lambda}{4}e_{367}^*,\\
\xi_{21} &= 1 + e_{147}^* + e_{257}^*,\\
\xi_{29} &= 1 + e_1e_2e_3e_4,\\
\xi_{42} &= 1, \text{ and}\\
\xi_{64} &= 0,
\end{aligned}$$

where $p \in k^*$ *and* $\lambda \in \mathbf{F}_q^* - (\mathbf{F}_q^*)^2$, *with* q *a power of any odd prime number. Then*

$$X(\mathbf{F}_q) = \bigcup_{\xi_i}(G\cdot\xi_i)(\mathbf{F}_q)$$

for all $\xi_i = \xi_0, \xi_1, \dots, \xi_{64}$ *and* $\xi_0', \xi_{10}', \xi_{20}'$.

3. The fixers

In this section we shall investigate the algebraic structure of the fixers in G. The results are summarized as

Theorem 3. *There exists a subgroup* H_i *of* $G^0_{\xi_i}$ *for* $i = 0, 1, 5, 10, 14, 20, 21, 29, 42$ *and a subgroup* H_i' *of* $G^0_{\xi_i'}$ *for* $i = 0, 10, 20$ *such that*

$$\begin{aligned}
&H_0 \cong G_2 \times G_2, \quad H_0' \cong G_2(\mathbf{F}_{q^2}), \quad H_1 \cong (GL_1 \times G_2)\cdot(G_a)^{14},\\
&H_5 \cong (GL_1 \times SL_2 \times_{\mathbf{Z}_2} SP_4)\cdot U^{19}, \quad H_{10} \cong (GL_1 \times SL_3 \times SL_3)\cdot(U^{21}),\\
&H_{10}' \cong (GL_1(\mathbf{F}_q) \times SL_3(\mathbf{F}_{q^2}))(U^{21}(\mathbf{F}_q)), \quad H_{14} \cong (GL_1 \times SL_4)\cdot(U^{26}),\\
&H_{20} \cong (GL_1 \times SL_6)\cdot(G_a^{12}), \quad H_{20}' \cong ((GL_1 \times SU_6)(\mathbf{F}_q))(G_a^{12}(\mathbf{F}_q)),\\
&H_{21} \cong (GL_1 \times SP_6 \times_{\mathbf{Z}_2} G_m)\cdot U^{26}, \quad H_{29} \cong (GL_1 \times SL_3 \times Spin_7)\cdot(U^{27}),\\
&H_{42} \cong (GL_7)\cdot(G_a^{21}).
\end{aligned}$$

Since each of the above cases has a similar proof, we shall only demonstrate the procedure for the case of $i = 0, 10, 20$ when the identity of the group G_{ξ_i} has two connected components.

H_0 is the subgroup of G generated by

$$T_0 = T \cap G_{\xi_0} = \left\{ t = \prod_{i=1}^{7} g_i(t_i);\ \prod_{i=1}^{6} t_i = t_1t_2t_3t_7 = 1,\right.$$
$$\left. t_4t_5t_6t_7 = 1,\ t_1, \dots, t_7 \in G_m \right\}$$

and by elements:

$$\begin{gathered}
s_{1,9}(\lambda),\ s_{2,8}(\lambda),\ s_{1,10}(\lambda),\ s_{3,8}(\lambda),\ s_{2,10}(\lambda),\ s_{3,9}(\lambda),\\
s_{4,12}(\lambda),\ s_{5,11}(\lambda),\ s_{4,13}(\lambda),\ s_{6,11}(\lambda),\ s_{5,13}(\lambda),\ s_{6,12}(\lambda),\\
s_{1,2}(\lambda)s_{7,10}(\lambda)s_{10,14}(\lambda),\ s_{8,9}(\lambda)s_{3,7}(\lambda)s_{3,14}(-\lambda),\\
s_{2,7}(\lambda)s_{8,10}(-\lambda)s_{2,14}(-\lambda),\ s_{9,14}(-\lambda)s_{1,3}(\lambda)s_{7,9}(-\lambda),\\
s_{2,3}(\lambda)s_{7,8}(\lambda)s_{8,14}(\lambda),\ s_{9,10}(\lambda)s_{1,14}(-\lambda)s_{1,7}(\lambda),\\
s_{6,7}(-\lambda)s_{6,14}(-\lambda)s_{11,12}(-\lambda),\ s_{13,14}(-\lambda)s_{7,13}(\lambda)s_{4,5}(-\lambda),\\
s_{5,7}(-\lambda)s_{5,14}(-\lambda)s_{11,13}(\lambda),\ s_{12,14}(\lambda)s_{7,12}(-\lambda)s_{4,6}(-\lambda),\\
s_{5,6}(-\lambda)s_{7,11}(\lambda)s_{11,14}(-\lambda),\ \text{and}\ s_{12,13}(-\lambda)s_{4,14}(-\lambda)s_{4,7}(-\lambda),
\end{gathered}$$

where $\lambda \in G_a$. One can easily verify that $H_0 \subset G^0_{\xi_0}$ by examining directly each of the elements listed above. Futhermore, by the lemma in §1, we have

$$\phi(H_0) = \left\{ \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \right\} \cong \{(A)\cdot(B)\} \cong G_2 \cdot G_2,$$

where

$$\alpha = \begin{bmatrix} & & & & & & a \\ & X & & & 0 & & b \\ & & & & & & c \\ & & & & & & d \\ & 0 & & & Y & & e \\ & & & & & & f \\ r & s & t & u & v & w & 1 \end{bmatrix}, \qquad X = (x_{ij}),\ Y = (y_{ij}) \in SL_3,$$

$$\beta = \begin{bmatrix} 0 & t & -s & & & & -a \\ -t & 0 & r & & 0 & & -b \\ s & -r & 0 & & & & -c \\ & & & 0 & w & -v & d \\ & 0 & & -w & 0 & u & e \\ & & & v & -u & 0 & f \\ a & b & c & -d & -e & -f & 0 \end{bmatrix},$$

$$\gamma = \begin{bmatrix} 0 & -c & b & & & & r \\ c & 0 & -a & & 0 & & s \\ -b & a & 0 & & & & t \\ & & & 0 & f & -e & -u \\ & 0 & & -f & 0 & d & -v \\ & & & e & -d & 0 & -w \\ -r & -s & -t & u & v & w & 0 \end{bmatrix},$$

$$\delta = \begin{bmatrix} x_{11}^{-1} & -x_{21} & -x_{31} & 0 & 0 & 0 & -r \\ -x_{12} & x_{22}^{-1} & -x_{32} & 0 & 0 & 0 & -s \\ -x_{13} & -x_{23} & x_{33}^{-1} & 0 & 0 & 0 & -t \\ 0 & 0 & 0 & y_{11}^{-1} & -y_{21} & -y_{31} & -u \\ 0 & 0 & 0 & -y_{12} & y_{22}^{-1} & -y_{32} & -v \\ 0 & 0 & 0 & -y_{13} & -y_{23} & y_{33}^{-1} & -w \\ -a & -b & -c & -d & -e & -f & 1 \end{bmatrix},$$

$$A = \begin{bmatrix} 0 & 2r & 2s & 2t & 2a & 2b & 2c \\ a & & & & 0 & t & -s \\ b & & X & & -t & 0 & r \\ c & & & & s & -r & 0 \\ r & 0 & -c & b & & & \\ s & c & 0 & -a & & -{}^tX & \\ t & -b & a & 0 & & & \end{bmatrix},$$

$$B = \begin{bmatrix} 0 & 2u & 2v & 2w & 2d & 2e & 2f \\ d & & & & 0 & w & -v \\ e & & Y & & -w & 0 & u \\ f & & & & v & -u & 0 \\ u & 0 & f & -e & & & \\ v & -f & 0 & d & & -{}^tY & \\ w & e & -d & 0 & & & \end{bmatrix}.$$

Since G_2 has no center, $H_0 \cong G_2 \times G_2$.

Similarly we can construct the elements that generate H_{10} and H_{20} and show that $G^0_{\xi_{10}} \supset H_{10} \cong (GL_1 \times SL_3 \times SL_3)\cdot(U^{21})$ and $G^0_{\xi_{20}} \supset H_{20} \cong (GL_1 \times SL_6)\cdot(G_a^{12})$.

Now let $g_0 = g_{10}h \in G(\mathbf{F}_{q^2})$ with $g_{10} \in H$ (see §2) such that

$$\phi(g_{10}) = \begin{bmatrix} \alpha & 0 \\ 0 & {}^t\alpha^{-1} \end{bmatrix}, \qquad \alpha = \begin{bmatrix} I_3 & I_3 & 0 \\ \mu I_3 & -\mu I_3 & 0 \\ 0 & 0 & -(2\mu)^{-3} \end{bmatrix},$$

i.e.,

$$\begin{aligned} g_{10} = &\{1 + \mu e_4 e_{7+1} + (e_1 - (\mu+1)e_4)e_{7+4} - (\mu+1)e_1e_4e_{7+1}e_{7+4}\} \\ &\cdot \{1 + \mu e_5 e_{7+2} + (e_2 - (\mu+1)e_5)e_{7+5} - (\mu+1)e_2e_5e_{7+2}e_{7+5}\} \\ &\cdot \{1 + \mu e_6 e_{7+3} + (e_3 - (\mu+1)e_6)e_{7+6} - (\mu+1)e_3e_6e_{7+3}e_{7+6}\} \\ &\cdot g_7(-(2\mu)^{-3}), \end{aligned}$$

where $\mu = \sqrt{\lambda}$, $\lambda \in \mathbf{F}_q^* - (\mathbf{F}_q^*)^2$, and $h \in H$ such that

$$\phi(h) = \begin{bmatrix} \alpha & 0 \\ 0 & {}^t\alpha^{-1} \end{bmatrix}, \qquad \alpha = \begin{bmatrix} I_5 & 0 \\ 0 & -I_2 \end{bmatrix},$$

i.e.,

$$h = g_1(1)g_2(1)g_3(1)g_4(1)g_5(1)g_6(-1)g_7(-1).$$

Let $g_{20} \in G(\mathbf{F}_{q^2})$ such that

$$g_{20} = s_{1,4}\left(-\frac{\sqrt{\mu^2+4}}{2}\right) s_{2,5}\left(-\frac{\sqrt{\mu^2+4}}{2}\right) s_{10,13}\left(-\frac{\sqrt{\mu^2+4}}{2}\right)$$
$$\cdot s_{8,11}(-\omega)s_{9,12}(-\omega)s_1((1-\omega^2)^{-1})s_{3,6}(\mu\omega^2(1-\omega^2)^{-1})$$
$$\cdot s_{1,4}(\omega(1-\omega^2))s_{2,5}(\omega(1-\omega^2)^{-1})s_{10,13}(-\mu)h',$$

where

$$\omega = \frac{\sqrt{\mu^2+4}-\mu}{2}$$

with $\mu = \sqrt{\lambda}$, $\lambda \in \mathbf{F}_q^* - (\mathbf{F}_q^*)^2$, and $h' \in H$ such that

$$\phi(h') = \begin{bmatrix} \alpha & 0 \\ 0 & {}^t\alpha^{-1} \end{bmatrix}, \qquad \alpha = \begin{bmatrix} I_5 & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \mu^{-1} \end{bmatrix},$$

i.e.,

$$h' = g_1(1)g_2(1)g_3(1)g_4(1)g_5(1)g_6(\mu)g_7(\mu^{-1}).$$

Then the action of g_i transforms ξ_i to ξ_i', i.e., $\rho(g_i)\cdot\xi_i = \xi_i'$ for $i = 0, 10, 20$. Hence for any $h_i \in H_i \subset G_{\xi_i}$ we have

$$\begin{aligned}\rho(g_ih_ig_i^{-1})\cdot\xi_i' &= \rho(g_ih_ig_i^{-1})\cdot\rho(g_i)\xi_i = \rho(g_ih_i)\cdot\rho(g_i)^{-1}\rho(g_i)\xi_i \\ &= \rho(g_ih_i)\cdot\xi_i = \rho(g_i)(\rho(h_i)\cdot\xi_i) = \rho(g_i)\cdot\xi_i = \xi_i'\end{aligned}$$

for $i = 0, 10, 20$. Hence $H_i' = g_iH_ig_i^{-1} \subset G_{\xi_i'}^0$ and

$$\begin{aligned}
H_0' &\cong \phi(H_0') = \phi(g_0)\phi(H_0)\phi(g_0)^{-1} \cong G_2(\mathbf{F}_{q^2}),\\
H_{10}' &\cong \phi(H_{10}') = \phi(g_{10})\phi(H_{10})\phi(g_{10})^{-1} \cong (GL_1(\mathbf{F}_q)\times SL_3(\mathbf{F}_{q^2}))(U^{21}(\mathbf{F}_q)),\\
H_{20}' &\cong \phi(H_{20}') = \phi(g_{20})\phi(H_{20})\phi(g_{20})^{-1} \cong ((GL_1\times SU_6)(\mathbf{F}_q))(G_a^{12}(\mathbf{F}_q)).
\end{aligned}$$

4. The classification

First we shall list some formulas for calculating the cardinalities of certain algebraic groups:

$$\begin{aligned}
|SL_n(\mathbf{F}_q)| &= q^{n^2-1}\prod_{1<i\le n}(1-q^{-i}),\\
|SU_n(\mathbf{F}_q)| &= q^{n^2-1}\prod_{1<i\le n}(1-(-q)^{-i}),\\
|SO_{2n+1}(\mathbf{F}_q)| &= |SP_{2n}(\mathbf{F}_q)| = q^{n(2n+1)}\prod_{1\le i\le n}(1-q^{-2i}),\\
|SO_{2n}(\mathbf{F}_q)| &= q^{n(2n-1)}\prod_{1\le i\le n-1}(1-q^{-2i})\cdot(1-q^{-n}),\\
|G_2(\mathbf{F}_q)| &= q^{14}(1-q^{-2})(1-q^{-6}),\\
|U^n(\mathbf{F}_q)| &= |G_a^n(\mathbf{F}_q)| = q^n,\\
|GL_1(\mathbf{F}_q)| &= |G_m(\mathbf{F}_q)| = q(1-q^{-1}),\\
|\mathrm{Spin}_n(\mathbf{F}_q)| &= |SO_n(\mathbf{F}_q)|.
\end{aligned}$$

By Theorem 2 in §2, we have

$$|X(\mathbf{F}_q)| \leq \sum_{\xi_i} |G \cdot \xi_i(\mathbf{F}_q)|$$

for $\xi_i = \xi_0, \xi_1, \dots, \xi_{64}$ and $\xi'_0, \xi'_{10}, \xi'_{20}$. Futhermore,

$$|G \cdot \xi_i(\mathbf{F}_q)| = \frac{|G(\mathbf{F}_q)|}{|G_{\xi_i}(\mathbf{F}_q)|}$$

for $\xi_i = \xi_1, \xi_5, \xi_{14}, \xi_{21}, \xi_{29}, \xi_{42}, \xi_{64}$, and

$$|G \cdot \xi_i(\mathbf{F}_q)| = \frac{|G(\mathbf{F}_q)|}{2|G_{\xi_i}(\mathbf{F}_q)|}$$

for $\xi_i = \xi_0, \xi'_0, \xi_{10}, \xi'_{10}, \xi_{20}, \xi'_{20}$.

By Theorem 3 in §3, we have $|G_{\xi_i}(\mathbf{F}_q)| \geq |H_i(\mathbf{F}_q)|$ for all ξ_i and ξ'_i. Put $(r) = 1 - q^{-r}$, and $(r)_+ = 1 + q^{-r}$. Then

$$\begin{aligned}
|G \cdot \xi_0(\mathbf{F}_q)| &\leq \frac{|G(\mathbf{F}_q)|}{2|H_0(\mathbf{F}_q)|} = \frac{|\mathrm{GSpin}_{14}(\mathbf{F}_q)|}{2|(G_2 \times G_2)(\mathbf{F}_q)|} \\
&= \frac{1}{2} q^{64}(1)(2)_+(6)_+(7)(8)(10), \\
|G \cdot \xi'_0(\mathbf{F}_q)| &\leq \frac{|G(\mathbf{F}_q)|}{2|H'_0(\mathbf{F}_q)|} = \frac{|\mathrm{GSpin}_{14}(\mathbf{F}_q)|}{2|G_2(\mathbf{F}_{q^2})|} \\
&= \frac{1}{2} q^{64}(1)(2)(6)(7)(8)(10), \\
G \cdot \xi_1(\mathbf{F}_q)| &\leq \frac{|G(\mathbf{F}_q)|}{|H_1(\mathbf{F}_q)|} = \frac{|\mathrm{GSpin}_{14}(\mathbf{F}_q)|}{|(GL_1 \times G_2)(\mathbf{F}_q) \cdot G_a^{14}(\mathbf{F}_q)|} \\
&= q^{63}(4)(7)(8)(10)(12), \\
|G \cdot \xi_5(\mathbf{F}_q)| &\leq \frac{|G(\mathbf{F}_q)|}{|H_5(\mathbf{F}_q)|} = \frac{|\mathrm{GSpin}_{14}(\mathbf{F}_q)|}{|(GL_1 \times SL_2 \times_{\mathbf{Z}_2} SP_4)(\mathbf{F}_q) \cdot U^{19}(\mathbf{F}_q)|} \\
&= q^{59}(2)_+(4)_+(6)(7)(10)(12), \\
|G \cdot \xi_{10}(\mathbf{F}_q)| &\leq \frac{|G(\mathbf{F}_q)|}{2|H_{10}(\mathbf{F}_q)|} = \frac{|\mathrm{GSpin}_{14}(\mathbf{F}_q)|}{2|(GL_1 \times SL_3 \times SL_3)(\mathbf{F}_q) \cdot U^{21}(\mathbf{F}_q)|} \\
&= \frac{1}{2} q^{54}(2)_+(3)_+^2(6)_+(7)(8)(10), \\
|G \cdot \xi'_{10}(\mathbf{F}_q)| &\leq \frac{|G(\mathbf{F}_q)|}{2|H'_{10}(\mathbf{F}_q)|} = \frac{|\mathrm{GSpin}_{14}(\mathbf{F}_q)|}{2|(GL_1(\mathbf{F}_q) \times SL_3(\mathbf{F}_{q^2})) \cdot U^{21}(\mathbf{F}_q)|} \\
&= \frac{1}{2} q^{54}(2)(7)(8)(10)(12),
\end{aligned}$$

$$|G\cdot\xi_{14}(\mathbf{F}_q)| \le \frac{|G(\mathbf{F}_q)|}{|H_{14}(\mathbf{F}_q)|} = \frac{|\mathrm{GSpin}_{14}(\mathbf{F}_q)|}{|GL_4(\mathbf{F}_q)\cdot U^{26}(\mathbf{F}_q)|}$$
$$= q^{50}(3)_+(7)(8)(10)(12),$$
$$|G\cdot\xi_{20}(\mathbf{F}_q)| \le \frac{|G(\mathbf{F}_q)|}{2|H_{20}(\mathbf{F}_q)|} = \frac{|\mathrm{GSpin}_{14}(\mathbf{F}_q)|}{2|(GL_1\times SL_6)(\mathbf{F}_q)\cdot G_a^{12}(\mathbf{F}_q)|}$$
$$= \frac{1}{2}q^{44}(3)_+(5)_+(6)_+(7)(8),$$
$$G\cdot\xi'_{20}(\mathbf{F}_q)| \le \frac{|G(\mathbf{F}_q)|}{|H'_{20}(\mathbf{F}_q)|} = \frac{|\mathrm{GSpin}_{14}(\mathbf{F}_q)|}{|(GL_1\times SU_6)(\mathbf{F}_q)\cdot G_a^{12}(\mathbf{F}_q)|}$$
$$= \frac{1}{2}q^{44}(3)(5)(6)_+(7)(8),$$

$$|G\cdot\xi_{21}(\mathbf{F}_q)| \le \frac{|G(\mathbf{F}_q)|}{|H_{21}(\mathbf{F}_q)|} = \frac{|\mathrm{GSpin}_{14}(\mathbf{F}_q)|}{|(GL_1\times SP_6\times_{\mathbf{Z}_2}G_m)(\mathbf{F}_q)\cdot U^{26}(\mathbf{F}_q)|}$$
$$= q^{43}(1)_+(2)_+(4)_+(7)(10)(12),$$
$$|G\cdot\xi_{29}(\mathbf{F}_q)| \le \frac{|G(\mathbf{F}_q)|}{|H_{29}(\mathbf{F}_q)|} = \frac{|\mathrm{GSpin}_{14}(\mathbf{F}_q)|}{|(GL_1\times SL_3\times \mathrm{Spin}_7)(\mathbf{F}_q)\cdot U^{27}(\mathbf{F}_q)|}$$
$$= q^{35}(2)_+(3)_+(4)_+(6)_+(7)(10),$$
$$|G\cdot\xi_{42}(\mathbf{F}_q)| \le \frac{|G(\mathbf{F}_q)|}{|H_{42}(\mathbf{F}_q)|} = \frac{|\mathrm{GSpin}_{14}(\mathbf{F}_q)|}{|GL_7(\mathbf{F}_q)\cdot G_a^{21}(\mathbf{F}_q)|} = q^{22}(3)_+(5)_+(6)_+(8),$$
$$|G\cdot\xi_{64}(\mathbf{F}_q)| = \frac{|G(\mathbf{F}_q)|}{|G_{\xi_{64}}(\mathbf{F}_q)|} = \frac{|\mathrm{GSpin}_{14}(\mathbf{F}_q)|}{|\mathrm{GSpin}_{14}(\mathbf{F}_q)|} = 1.$$

Adding these together, we get

$$\sum_{\xi_i}|G\cdot\xi_i(\mathbf{F}_q)| \le q^{64} = |X(\mathbf{F}_q)|.$$

Hence we verified that

$$\sum_{\xi_i}|G\cdot\xi_i(\mathbf{F}_q)| = q^{64} = |X(\mathbf{F}_q)|.$$

Since the G-orbits $G\cdot\xi_i$ for all ξ_i are distinct and every G-orbit in X has an $\mathbf{F}_q$-rational point at least for a high power q of any prime number, we conclude that there are no other G-orbits in X, hence

$$X = \coprod_{\xi_i} G\cdot\xi_i,$$

and $G_{\xi_i} = H_i$ for all ξ_i. We summarize with

Theorem 4. *If q is a power of any odd prime, then*

$$X(\mathbf{F}_q) = \coprod_i G\cdot\xi_i(\mathbf{F}_q), \quad \mathrm{codim}(G\cdot\xi_i) = i,$$

for all ξ_i in Table 1, *where $G_{\xi_i}(\mathbf{F}_q)$ denotes the fixer of spinor ξ_i with respect to $G(\mathbf{F}_q)$ and $G^0_{\xi_i}(\mathbf{F}_q)$ denotes the connected component of the identity of the group $G_{\xi_i}(\mathbf{F}_q)$.*

TABLE 1

i	ξ_i	$G^0_{\xi_i}(\mathbf{F}_q)$	$[G_{\xi_i}(\mathbf{F}_q) : G^0_{\xi_i}(\mathbf{F}_q)]$
0	$\mu(1 + e_1e_2e_3e_7 + e_4e_5e_6e_7 + e_7^*)$, $\mu \in \mathbf{F}_q^*$	$(G_2 \times G_2)(\mathbf{F}_q)$	2
0	$\mu(1 + 4^{-1}e_{123}^* + (4\lambda)^{-1}(e_{156}^* - e_{246}^* + e_{345}^*) - e_7^*)$, $\mu \in \mathbf{F}_q^*$, $\lambda \in \mathbf{F}_q^* - (\mathbf{F}_q^*)^2$	$G_2(\mathbf{F}_{q^2})$	2
1	$1 + e_1e_2e_3e_7 + e_4e_3e_5e_7 + e_6e_5e_2e_7 + e_7^*$	$(GL_1 \times G_2)(\mathbf{F}_q) \cdot G_a^{14}(\mathbf{F}_q)$	1
5	$1 + e_1e_2e_3e_7 + e_3e_4e_5e_7 + e_7^*$	$(GL_1 \times SL_2 \times_{\mathbf{Z}_2} SP_4)(\mathbf{F}_q) \cdot U^{19}(\mathbf{F}_q)$	1
10	$1 + e_1e_2e_3e_7 + e_7^*$	$(GL_1 \times SL_3 \times SL_3)(\mathbf{F}_q) \cdot U^{21}(\mathbf{F}_q)$	2
10	$1 + 4^{-1}e_{123}^* + (4\lambda)^{-1}(e_{156}^* - e_{246}^* + e_{345}^*)$, $\lambda \in \mathbf{F}_q^* - (\mathbf{F}_q^*)^2$	$(GL_1(\mathbf{F}_q) \times SL_3(\mathbf{F}_{q^2})) \cdot U^{21}(\mathbf{F}_q)$	2
14	$1 + e_1e_2e_3e_7 + e_{147}^* + e_{257}^*$	$GL_4(\mathbf{F}_q) \cdot U^{26}(\mathbf{F}_q)$	1
20	$1 + e_7^*$	$(GL_1 \times SL_6)(\mathbf{F}_q) \cdot G_a^{12}(\mathbf{F}_q)$	2
20	$1 + e_{147}^* + e_{257}^* - 4^{-1}\lambda e_{367}^*$, $\lambda \in \mathbf{F}_q^* - (\mathbf{F}_q^*)^2$	$(GL_1 \times SU_6)(\mathbf{F}_q) \cdot G_a^{12}(\mathbf{F}_q)$	2
21	$1 + e_{147}^* + e_{257}^*$	$(GL_1 \times SP_6 \times_{\mathbf{Z}_2} G_m)(\mathbf{F}_q) \cdot U^{26}(\mathbf{F}_q)$	1
29	$1 + e_1e_2e_3e_4$	$(GL_1 \times SL_3 \times \mathrm{Spin}_7)(\mathbf{F}_q) \cdot U^{27}(\mathbf{F}_q)$	1
42	1	$GL_7(\mathbf{F}_q) \cdot G_a^{21}(\mathbf{F}_q)$	1
64	0	$\mathrm{GSpin}_{14}(\mathbf{F}_q)$	1

REFERENCES

1. M. F. Atiyah, *Resolution of singularities and division of distribution*, Comm. Pure Appl. Math. **23** (1970), 145–150.
2. I. N. Bernstein and S. I. Gel'fand, *Meromorphic property of the function* P^λ, Functional Anal. Appl. **3** (1969), 68–69.
3. J. Denef, *On the degree of Igusa's local zeta functions*, Amer. J. Math. **109** (1987), 991–1008.
4. J. Igusa, *A classification of spinors up to dimension twelve*, Amer. J. Math. **92** (1970), 997–1028.
5. ——, *On the arithmetic of Pfaffians*, Nagoya Math. J. **47** (1972), 169–198.
6. ——, *Complex powers and asymptotic expansion*. I, Crelles J. Math. **268/269** (1974), 110–130; II ibid. **278/279** (1975), 307–321.
7. ——, *Exponential sums associated with a Freudenthal quartic*, J. Fac. Sci. Univ. Tokyo Sect. I A Math. **24** (1977), 231–246.
8. ——, *Some results on p-adic complex powers*, Amer. J. Math. **106** (1984), 1013–1032.
9. ——, *On functional equations of complex powers*, Invent. Math. **85** (1986), 1–29.
10. ——, *Zeta distributions associated with some invariants*, Amer. J. Math. **109** (1987), 1–34.
11. ——, *On the arithmetic of a singular invariant*, Amer. J. Math. **110** (1988), 197–233.
12. ——, *b-functions and p-adic integrals*, Algebraic Anal. **1** (1988), 231–241.
13. V. G. Kac and È. B. Vinberg, *Spinors of* 13*-dimensional spaces*, Adv. in Math. **30** (1978), 137–155.
14. M. Kashiwara, *b-functions and holonomic systems*, Invent. Math. **38** (1976), 33–53.
15. T. Kimura, *The b-functions and holonomy diagrams of irreducible regular prehomogeneous vector spaces*, Nagoya Math. J. **85** (1982), 1–80.
16. T. Kimura, F. Sato, and X.-W. Zhu, *On the poles of p-adic complex powers and the b-functions of prehomogeneous vector spaces*, Amer. J. Math. **112** (1990), 423–437.
17. F. Loeser, *Fonction d'Igusa p-adiques et polynômes de Bernstein*, Amer. J. Math. **110** (1988), 1-21.
18. D. Meuser, *On the poles of a local zeta function for the curves*, Invent. Math. **73** (1983), 445–465.
19. ——, *The meromorphic continuation of a zeta function of Weil and Igusa type*, Invent. Math. **85** (1986), 493–514.
20. V. L. Popov, *Classification of spinors of dimension fourteen*, Trans. Moscow Math. Soc. **1** (1980), 181–232.
21. M. Sato and T. Kimura, *A classification of irreducible prehomogeneous vector spaces and their relative invariants*, Nagoya Math. J. **65** (1977), 1–155.
22. M. Sato and T. Shintani, *On zeta function associated with prehomogeneous vector spaces*, Ann. Math. **100** (1974), 131–170.
23. J.-P. Serre, *Méthodes adéliques*, Résumé des cours et travaux, Collége de France 1981–1982, pp. 81–89.

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